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On Nevai's Characterization of Measures with Almost Everywhere Positive Derivative

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It is known that if $d\mu$ is a finite positive Borel measure on the unit circle $\partial\Delta := \{z \in \mathbf{C} : |z| = 1\}$ with $\mu' > 0$ a.e. in $[0, 2\pi]$, then

$$\lim_{n \to \infty} \int_0^{2\pi} \left| \frac{|\varphi_n(z)|^2}{|\varphi_{n-l}(z)|^2} - 1 \right| d\theta = 0, \qquad z = e^{i\theta} \qquad (*)$$

uniformly in $l \geq 0$, where $\{\varphi_n(z)\}_{n=0}^{\infty}$ is the set of orthonormal polynomials with respect to $d\mu$. Recently, Nevai proved that if $(*)$ holds uniformly in $l \geq 0$, then $\mu' > 0$ a.e. in $[0, 2\pi]$. In this note we establish another characterization of such measures in terms of Turán-type inequalities and we utilize it to give an alternative proof of Nevai's result.

Let $d\mu$ be a finite positive Borel measure on the unit circle $\partial\Delta := \{z \in \mathbf{C} : |z| = 1\}$ with its support an infinite set. Let $\varphi_n(z) = \varphi_n(d\mu, z) := \kappa_n z^n + \cdots \in \mathscr{P}_n$, $\kappa_n > 0$, $n = 0, 1, 2, \ldots$, be the nth orthonormal polynomial with respect to $d\mu$; that is,

$$\frac{1}{2\pi} \int_{\partial\Delta} \varphi_m(z) \overline{\varphi_n(z)} \, d\mu = \delta_{m,n}, \qquad m, n = 0, 1, 2, \ldots.$$

Rahmanov proved (cf. [9, p. 106])

THEOREM A. *If $\mu' > 0$ a.e. in $[0, 2\pi]$, then*

$$\lim_{n \to \infty} \frac{z\varphi_n(z)}{\varphi_{n+1}(z)} = 1 \qquad (1)$$

uniformly for $|z| \geq 1$.

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In [5] (see also [8]), Máté *et al.* gave a much simpler proof of Theorem A. In doing so, they established (cf. [5, Theorem 3, p. 64])

THEOREM B. *If $\mu' > 0$ a.e. in $[0, 2\pi]$, then*

$$\lim_{n \to \infty} \int_0^{2\pi} \left| \frac{|\varphi_n(z)|^2}{|\varphi_{n+l}(z)|^2} - 1 \right| d\theta = 0, \qquad z = e^{i\theta}, \tag{2}$$

uniformly in $l \geq 0$.

On the one hand, Theorem A follows from Theorem B (cf. [5, Theorem 2, p. 64]). On the other hand, as noted in [5, Remark (a), p. 64], Theorem B follows from Theorem A except for the uniformity in l. In fact, recently Nevai (cf. [7, Theorem 1.1, p. 295]) proved that (2) implies $\mu' > 0$ a.e. in $[0, 2\pi]$. In this paper we establish another characterization of measures with positive derivative and we use it to give an alternative proof of the above mentioned result of Nevai.

In [4], the following inequality of Turán played an essential role.

THEOREM C (cf. [6, Corollary 7.5, p. 258]). *If $\mu' > 0$ a.e. in $[0, 2\pi]$, then for every $\delta \in (0, 2\pi]$ there is an $\varepsilon > 0$ such that*

$$\int_E |\varphi_n|^2 \, d\mu \geq \varepsilon \tag{3}$$

holds for every $n \geq 0$ and every Borel set $E \subset \partial\Delta$ with Lebesgue measure $|E| \geq \delta$.

We are concerned with the following question: Is $\mu' > 0$ a.e. in $[0, 2\pi]$ necessary for (3)? To answer this question, we prove the following refinement of Theorem C which characterizes measures with positive derivative in terms of Turán-type inequalities.

THEOREM 1. *Let $d\mu$ be a finite positive Borel measure on $\partial\Delta$ with infinite support. Then $\mu' > 0$ a.e. in $[0, 2\pi]$ if and only if for every $\delta \in (0, 2\pi]$ and $\rho \in (0, 1)$, there exists an integer $N(\delta, \rho)$ such that*

$$\int_E |\varphi_n|^2 \, d\mu \geq \rho\delta, \qquad n \geq N(\delta, \rho), \tag{4}$$

for all Borel sets $E \subset \partial\Delta$ with $|E| \geq \delta$.

Proof. Let us first assume that $\mu' > 0$ a.e. in $[0, 2\pi]$. Then by [6, Theorem 7.4, p. 257],

$$\lim_{n \to \infty} \int_E |\varphi_n|^2 \, d\mu = |E|$$

uniformly for all Borel sets $E \subset \partial\Delta$. Thus, given $\delta \in (0, 2\pi]$ and $\rho \in (0, 1)$, we can find an integer $N(\delta, \rho)$ such that

$$\left| \int_E |\varphi_n|^2 \, d\mu - |E| \right| < (1-\rho)\delta, \qquad n \geqslant N(\delta, \rho),$$

uniformly for all Borel sets $E \subset \partial\Delta$. Hence, when $n \geqslant N(\delta, \rho)$,

$$\int_E |\varphi_n|^2 \, d\mu \geqslant |E| - (1-\rho)\delta \geqslant \rho\delta$$

for all Borel sets $E \subset \partial\Delta$ with $|E| \geqslant \delta$, which establishes (4).

Now assume that (4) holds. From the Lebesgue decomposition (cf. [10, p. 21])

$$d\mu = \mu' \, d\theta + d\mu_s,$$

where $d\mu_s \perp d\theta$. Assume that $d\mu_s$ is concentrated on a Borel set A. Then

$$|A| = 0.$$

Given $\varepsilon > 0$, take $\delta = 2\pi$ and $\rho \in (0, 1)$ such that $2\pi - \rho\delta = 2\pi(1-\rho) < \varepsilon$. Then, for $n \geqslant N(2\pi, \rho)$, we get from (4) with $E = A^c$

$$\int_A |\varphi_n|^2 \, d\mu = \int_0^{2\pi} |\varphi_n|^2 \, d\mu - \int_{A^c} |\varphi_n|^2 \, d\mu \leqslant 2\pi - \rho\delta < \varepsilon.$$

Note that

$$\int_0^{2\pi} |\varphi_n|^2 \, d\mu_s = \int_A |\varphi_n|^2 \, d\mu_s = \int_A |\varphi_n|^2 \, d\mu,$$

and so

$$\int_0^{2\pi} |\varphi_n|^2 \, d\mu_s < \varepsilon,$$

when $n \geqslant N(2\pi, \rho)$. Hence,

$$\lim_{n \to \infty} \int_0^{2\pi} |\varphi_n|^2 \, d\mu_s = 0. \tag{5}$$

Now set

$$B := \{\theta \in [0, 2\pi] : \mu'(\theta) = 0\}.$$

If $|B| =: \delta_0 > 0$, then for $n \geqslant N(\delta_0, \frac{1}{2})$, we have from (4)

$$\frac{1}{2}\delta_0 \leqslant \int_B |\varphi_n|^2 \, d\mu = \int_B |\varphi_n|^2 \, d\mu_s \leqslant \int_0^{2\pi} |\varphi_n|^2 \, d\mu_s,$$

which contradicts (5). Therefore $|B| = 0$, i.e., $\mu' > 0$ a.e. in $[0, 2\pi]$. ∎

Now we state and prove Nevai's result (cf. [7, Theorem 1.1, p. 295]).

THEOREM 2. *Let $d\mu$ be a finite positive Borel measure on $\partial\Delta$ with infinite support. Then $\mu' > 0$ a.e. in $[0, 2\pi]$ if and only if* (2) *holds uniformly in $l \geqslant 0$.*

In the proof of Theorem 2 we make use of the following lemma in [6, Lemma 4.2, p. 248]. It is a consequence of [2, Theorem 5.2.2, p. 198] (also see [1, formula (1.20), p. 7]).

LEMMA 3. *For any $f \in C(\partial\Delta)$,*

$$\lim_{n \to \infty} \int_0^{2\pi} \frac{f(z)}{|\varphi_n(z)|^2} \, d\theta = \int_0^{2\pi} f(z) \, d\mu, \qquad z = e^{i\theta}.$$

Proof of Theorem 2. In view of Theorem B, we only need show that (2) implies $\mu' > 0$ a.e. in $[0, 2\pi]$. To do so, it suffices, from Theorem 1 to show that (2) implies (4).

Assume that (2) holds. Let $\delta \in (0, 2\pi]$ and $\rho \in (0, 1)$ be given. We will proceed in three steps.

Step I. We first show that (4) holds uniformly for every set E which is a finite union of open intervals.

Assume that

$$E = \bigcup_{i=1}^{m} (\alpha_i, \beta_i) \subset \partial\Delta, \qquad (\alpha_i, \beta_i) \cap (\alpha_j, \beta_j) = \varnothing, \qquad i \neq j, \tag{6}$$

and $|E| \geqslant \delta$.

It is easy to see that there exists an $f_{E,\rho} \in C(\partial\Delta)$ and $E_1 := \bigcup_{i=1}^{m} (\alpha_i', \beta_i') \subset E$ such that

$$\chi_{E_1}(z) \leqslant f_{E,\rho}(z) \leqslant \chi_E(z), \qquad z \in \partial\Delta$$

(where χ_K denotes the characteristic function of the set $K \subset \mathbf{C}$), and

$$|E_1| = \rho^{1/2} \, |E|.$$

By (2), we can find an integer $M(\delta, \rho)$ such that, for $n \geqslant M(\delta, \rho)$,

$$\int_0^{2\pi} f_{E,\rho}(z) \left| \frac{|\varphi_n(z)|^2}{|\varphi_{n+l}(z)|^2} - 1 \right| d\theta \leqslant \int_0^{2\pi} \left| \frac{|\varphi_n|^2}{|\varphi_{n+l}|^2} - 1 \right| d\theta$$
$$< (\rho^{1/2} - \rho)\delta$$

uniformly in $l \geqslant 0$ and E is of the form given in (6). Thus, for $n \geqslant M(\delta, \rho)$,

$$\int_0^{2\pi} f_{E,\rho} \frac{|\varphi_n|^2}{|\varphi_{n+l}|^2} d\theta \geqslant \int_0^{2\pi} f_{E,\rho}\, d\theta - (\rho^{1/2} - \rho)\delta$$
$$\geqslant \rho^{1/2} \int_0^{2\pi} f_{E,\rho}\, d\theta \geqslant \rho^{1/2} \int_{E'} d\theta = \rho\, |E|$$

uniformly in $l \geqslant 0$ and E of the form given in (6) with $|E| \geqslant \delta$. From Lemma 3, by letting $l \to \infty$, we get

$$\int_0^{2\pi} f_{E,\rho}\, |\varphi_n|^2\, d\mu \geqslant \rho\, |E|, \qquad n \geqslant M(\delta, \rho).$$

Hence,

$$\int_E |\varphi_n|^2\, d\mu \geqslant \int_0^{2\pi} f_{E,\rho}\, |\varphi_n|^2\, d\mu \geqslant \rho\, |E|, \qquad n \geqslant M(\delta, \rho),$$

uniformly for E of the form given in (6) with $|E| \geqslant \delta$.

Step II. Next we show that (4) holds uniformly for every open subset E of $\partial\Delta$ with $|E| \geqslant \delta$.

We can write such an open set as

$$E = \bigcup_{i=1}^{\infty} (\alpha_i, \beta_i), \qquad (\alpha_i, \beta_i) \cap (\alpha_j, \beta_j) = \varnothing, \qquad i \neq j,$$

and

$$|E| = \sum_{i=1}^{\infty} (\beta_i - \alpha_i) \geqslant \delta.$$

Let m be so large that

$$\sum_{i=1}^{m} (\beta_i - \alpha_i) \geqslant \rho^{1/2}\delta\ .$$

Then, when $n \geqslant M(\rho^{1/2}\delta, \rho^{1/2})$, where $M(\cdot, \cdot)$ is given by Step I, we have

$$\int_E |\varphi_n|^2 \, d\mu \geqslant \int_{\bigcup_{i=1}^m (\alpha_i, \beta_i)} |\varphi_n|^2 \, d\mu \geqslant \rho^{1/2} \cdot \rho^{1/2}\delta = \rho\delta$$

uniformly for any open set $E \subset \partial\Delta$ with $|E| \geqslant \delta$.

Step III. We complete the proof by showing that E can be any Borel set with $|E| \geqslant \delta$.

For such a set E we can find, by the outer-regularity of measure $|\varphi_n|^2 \, d\mu$, an open set $\mathcal{C}_{n,E} \supset E$ such that

$$\rho^{-1/2} \int_E |\varphi_n|^2 \, d\mu \geqslant \int_{\mathcal{C}_{n,E}} |\varphi_n|^2 \, d\mu.$$

Therefore, by Step II and the inequality $|\mathcal{C}_{n,E}| \geqslant \delta$, we have

$$\int_E |\varphi_n|^2 \, d\mu \geqslant \rho^{1/2} \int_{\mathcal{C}_{n,E}} |\varphi_n|^2 \, d\mu \geqslant \rho^{1/2} \cdot \rho^{1/2}\delta = \rho\delta,$$

when $n \geqslant M(\rho^{1/4}\delta, \rho^{1/4})$. ∎

Finally, we remark that in [3], Lubinsky gave an explicit expression of a pure jump distribution (i.e., a discrete measure) $d\nu$ on $\partial\Delta$ such that (cf. [3, formula (2.22), p. 523])

$$\sum_{j=2 \cdot 3^{l-1}+1}^{2 \cdot 3^l} |\Phi_j(0)|^2 \leqslant C\left(\frac{\log l}{l}\right),$$

l large enough, where $\Phi_j(z) := (1/\kappa_j)\varphi_j(z)$ and $C > 0$ is a constant independent of l. Consequently,

$$\lim_{n \to \infty} |\Phi_n(0)| = 0,$$

which is equivalent to (1) (cf., e.g., [5, formula (7), p. 66]). Summarizing these remarks, we can state

PROPOSITION 4. *There exists a finite positive Borel measure that is a pure jump distribution but for which* (1) *holds.*

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